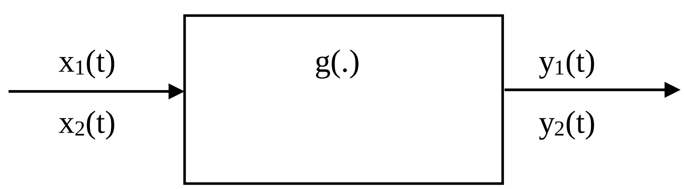
Chapter 1

Linear Systems, Voltage and Operational Amplifiers

1. Introduction

2. Linear Systems

A linear system is usually described by a set of linear equations. A linear system must satisfy superposition.



where the responses $y_1(t) = g(x_1(t))$ and $y_2(t) = g(x_2(t))$ are the outputs of the system given the inputs $x_1(t)$ and $x_2(t)$, respectively. Superposition then leads to the following:

If the input to the system is a linear combination of the individual inputs, $a_1x_1(t) + a_2x_2(t)$, then the output is the same linear combination to their corresponding outputs

$$a_1x_1(t)+a_2x_2(t) \Rightarrow a_1y_1(t)+a_2y_2(t)$$

Note that superposition applies only to the zero-state response (response obtained under the assumption that the initial conditions are zero). A linear system is composed of linear components only.

Except in some cases, most of the topics discussed in this course will be based on the analysis and design of linear systems. Superposition will hence be a major tool in simplifying the analysis process.

2.1 Total Harmonic Distortion

If the input to a nonlinear system is a sinewave at a frequency $\omega = \omega_0$, then the output is

$$v(t) = V_O + V_1 \sin(\omega_o t + \phi_1) + V_2 \sin(2\omega_o t + \phi_2) + V_3 \sin(3\omega_o t + \phi_3) + \ldots$$

We can see that the output is made of a DC component, the fundamental which is the desired component, and multiple harmonics (in general an infinite number).

$$THD = 100\% \times \frac{\sqrt{\sum_{i=2}^{\infty} V_i^2}}{V_1}$$

It is clear that if the system were linear, only the sinewave at ω_0 may appear at the output.

Example 2.1

Show that the system described by $y = x^2$ is not a linear system

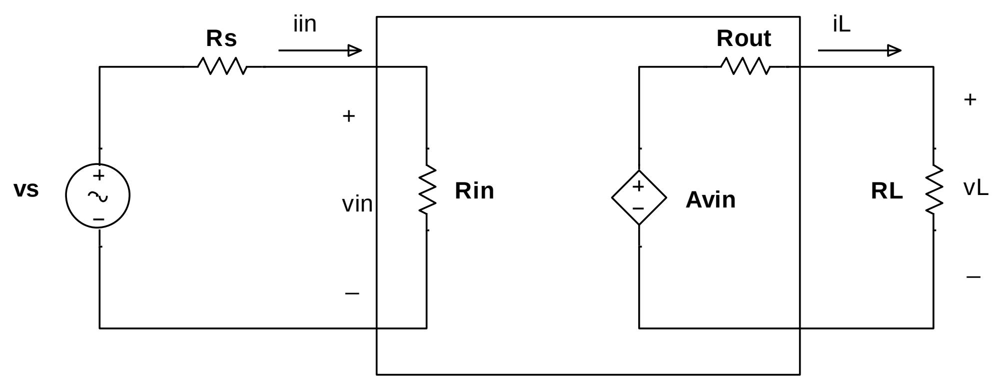
$y_1 = a_1^2\ x_1^2$, $y_2 = a_2^2\ x_2^2$, but $(a_1x_1 + a_2x_2)^2 = a_1^2\ x_1^2 + a_2^2\ x_2^2 + 2\ a_1x_1\ a_2x_2 \neq a_1^2\ x_1^2 + a_2^2\ x_2^2$

if $x(t) = A \cos\omega_0 t$, then $y(t) = A^2 \cos^2\omega_0 t = A^2/2 + A^2/2\cos 2\omega_0 t$

We can see that, not only did we lose the desired component, but now we have 2 unwanted terms, the DC component and the second harmonic.

3. Voltage Amplifier

A voltage amplifier is a system described by the block shown below



The voltage amplifier block is preceded by the combination (v_s, R_s), which could represent a voltage source and its internal resistance, or the Thevenin equivalent of a previous stage. The amplifier is terminated with a resistor R_L, which could be a load resistor, or the input resistance of a subsequent stage. v_{in} is the voltage at the input of the amplifier, and v_L is the voltage across the load.

Using voltage division at the input loop and the output loop, we obtain:

$$v_{in} = \frac{R_{in}}{R_{in} + R_s} v_s \quad and \quad v_L = \frac{R_L}{R_L + R_{out}} A v_{in}$$

Hence

$$v_L = A \frac{R_L}{R_L + R_{out}} \frac{R_{in}}{R_{in} + R_s} v_s, \quad A_v = \frac{v_L}{v_s} = A \frac{R_L}{R_L + R_{out}} \frac{R_{in}}{R_{in} + R_s}$$

Since the designer has no control over R_s or R_L, and he needs to amplify v_s (the open circuit voltage for the left source) so that $v_L = Av_s$, from a practical point of view, the characteristics of the amplifier should be such that

Practical voltage amplifier $R_{in} >> R_s$, and $R_{out} << R_L$

Ideal voltage amplifier, $R_{in} = \infty$, $R_{out} = 0$, and $A_v = A$

One can easily show that

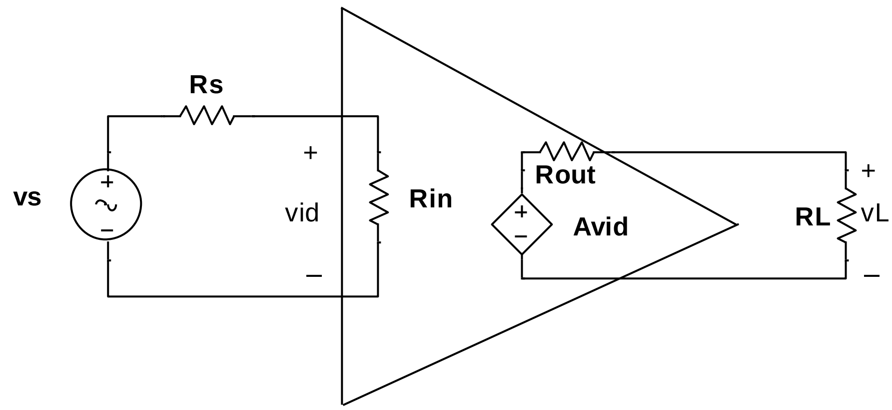
$$|A_i| = |A_v| \frac{R_s + R_{in}}{R_L}$$

Certainly, if $R_{in} = \infty$, $i_{in} = 0$, and hence the current gain $|Ai| = \infty$.

4. Operational Amplifier (Op Amp) with Source and Load

Op Amps are integrated circuits (ICs) made of hundreds of transistors with the task of amplifying the difference of two input signals. Hence the input stage is a differential pair, with a very high input resistance (megohm range), and the output has a low resistance so that the current delivered by the op amp becomes independent of the load. The high input resistance and the low output resistance makes this device most appropriate for voltage amplification and many other analog applications.

We will see in the few applications presented that it helps perform operations, be they arithmetic or otherwise, hence the label operational amplifier. In addition, we will understand why the open loop gain has been made to be a very large number, usually in the hundreds of thousands, or higher. Because of DC coupling, this amplifier can operate at frequencies all the way down to DC. However, its slew rate, a measure of how fast the output voltage varies with time, is not a very large number. In the case of internal compensation, a small feedback capacitor inside the op amp is used to improve the stability of the amplifier, while minimizing the generation of oscillations and ringing at the output. It is this capacitor that keeps the op amp from allowing the slew rate to be a higher number, or the op amp from reacting faster to voltage changes.



Op Amp Characteristics:

$v_+ = v_p$ = noninverting input

$v_- = v_n$ = inverting input

$v_{id} = v_+ - v_- = v_p - v_n$ = differential input (more about it in the differential amplifier topic)

R_{in} = input resistance

R_{out} = output resistance

A = open loop gain

(v_s, R_s) = input source voltage and its internal resistance or the Thevenin equivalent of the stage preceding the Op Amp

R_L = Load resistance or input resistance of the stage following the Op Amp.

Using the results obtained in the previous section, we have

$$v_{id} = \frac{R_{in}}{R_{in} + R_s} v_s \quad \text{and} \quad v_L = \frac{R_L}{R_L + R_{out}} A v_{id}$$

Hence

$$v_L = A \frac{R_L}{R_L + R_{out}} \frac{R_{in}}{R_{in} + R_s} v_s, \quad A_v = \frac{v_L}{v_s} = \frac{R_L}{R_L + R_{out}} \frac{R_{in}}{R_{in} + R_s}$$

Ideal Op Amps have the following characteristics $R_{in} = \infty$, $R_{out} = 0$, $A = \infty$

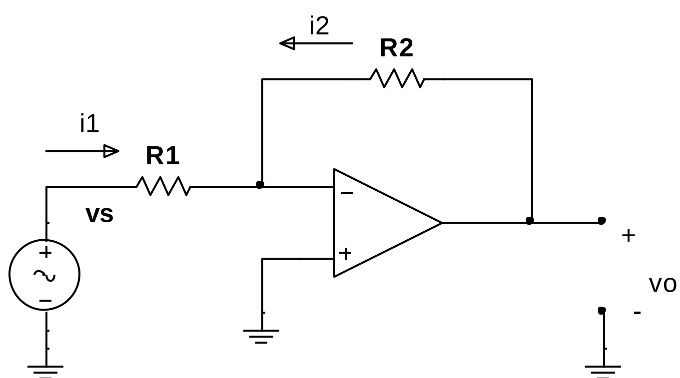
5. Op Amp Applications

One can consult material from publishers, industry or the internet, and one sees that there is almost no limit to what useful circuits you can create with operational amplifiers. We will analyze in this section the most popular that have been either investigated or utilized. All of the subsequent circuits in this topic will involve ideal Op Amps.

In addition, all circuits that are investigated in this topic will exhibit **negative feedback** (some connection between the output and the inverting input, either directly or through a component). In that case, the following will prevail:

- Since we are working with ideal Op Amps, and $R_{in} = \infty$, any current entering the Op Amp is zero, and since $\mathbf{A = \infty}$, and **the output is finite**, one of the inputs will track the other inputs until they become equal to each other, $\mathbf{v_{id} = 0}$

5.1. Inverting Amplifier



- The noninverting input is physically connected to ground
- Due to negative feedback, $v_{id} = 0$, hence the inverting input will have zero potential, and will be labeled a virtual ground

5.1.1. Voltage Gain

$$-v_s + R_1 i_1 = 0$$

$$-v_o + R_2 i_2 = 0$$

$$i_1 + i_2 = 0$$

$$A_v = \frac{v_o}{v_s} = -\frac{R_2}{R_1}$$

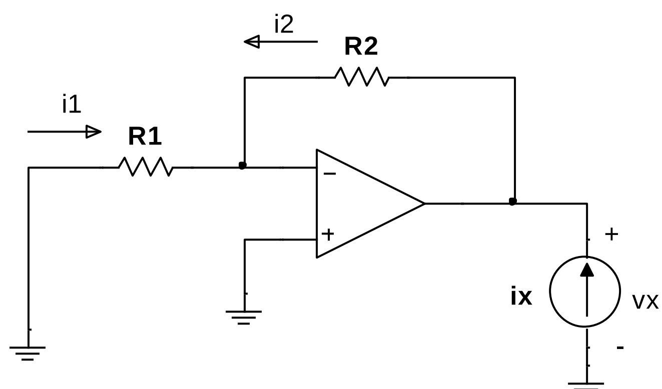
Note the 180 degrees in phase difference between the input and the output signals

5.1.2. Input Resistance

$$R_{in} = \frac{v_o}{v_s} = R_1$$

5.1.3. Output Resistance

In the laboratory, you have learned to measure the output resistance by measuring the open-circuit voltage, then loading the device with a variable resistor box and changing the resistance until the output voltage became half of the open-circuit voltage. In that case, the output resistance of the device is the resistance read on the resistor box.
On paper, we reduce the independent power supplies to zero (replace the voltage sources by a short circuit, and the current sources by an open circuit). The Thevenin equivalent of the remaining circuit is just a resistance. We then apply to the output a test current source, current i_x, and then evaluate the voltage across it, v_x. The ratio of v_x over i_x is the desired output resistance.



$$-v_x + R_2 i_2 = 0$$
$$R_1 i_1 = 0$$
$$i_2 = -i_1 = 0$$
$$hence \;\; v_x = 0$$
$$R_{out} = \frac{v_x}{i_x} = 0$$

We can see that the inverting amplifier is not the ideal voltage amplifier since $R_{in} \neq \infty$. Otherwise the output resistance is ideal, and the gain can be controlled by external components whose tolerances can be very low. If we purchase resistors with 1% tolerance, then the gain will have a 2% tolerance, which is absolutely acceptable.

One can show that as long as the open loop gain is a very large number, the voltage gain depends only on the external component. This is the reason we require that the Op Amps have a very large open loop gain so that you can choose any Op Amp regardless of its open loop gain or manufacturer, as long as it is a very large number so that the voltage gain can be controlled by external component.

Example

Problem:

Design an inverting amplifier with a gain of 20 dB, and an input resistance of 50 kΩ

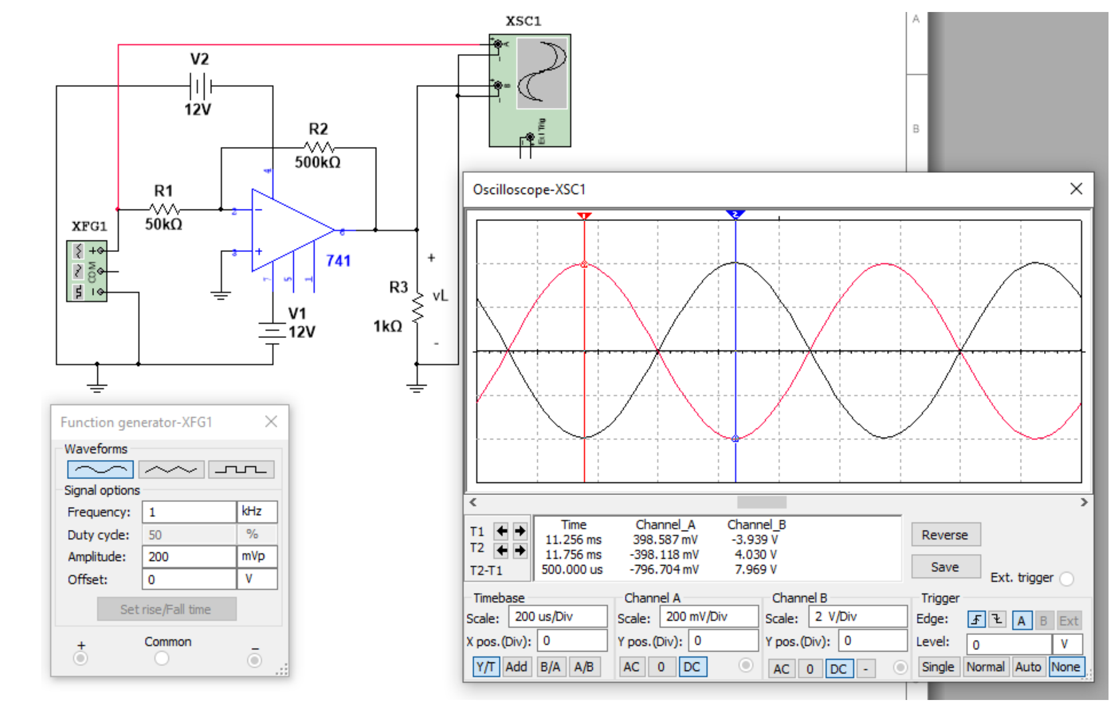
Solution:

Since $G_{dB} = 20 \log_{10} G$, $G = 10^{G_{db}/20}$. Hence $G = 10$

$|A_v| = R_2/R_1 = 10$

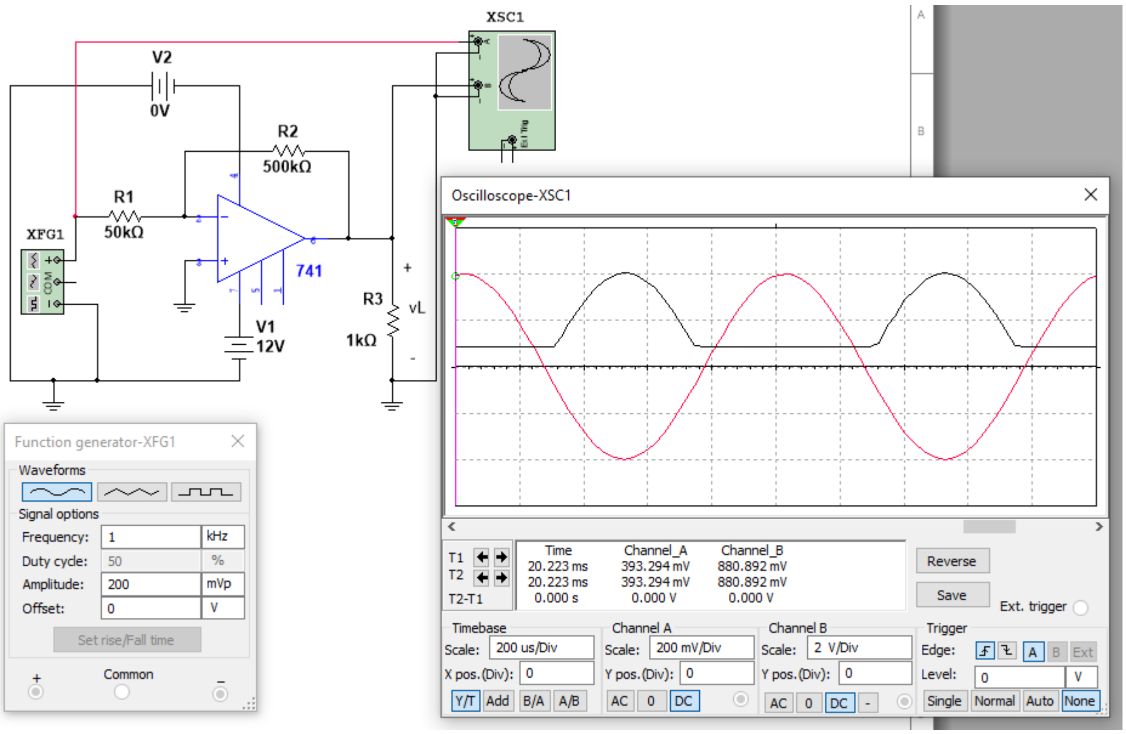
$R_2 = 10R_1$, and since $R_{in} = R_1 = 50$ kΩ

$R_2 = 500$ kΩ

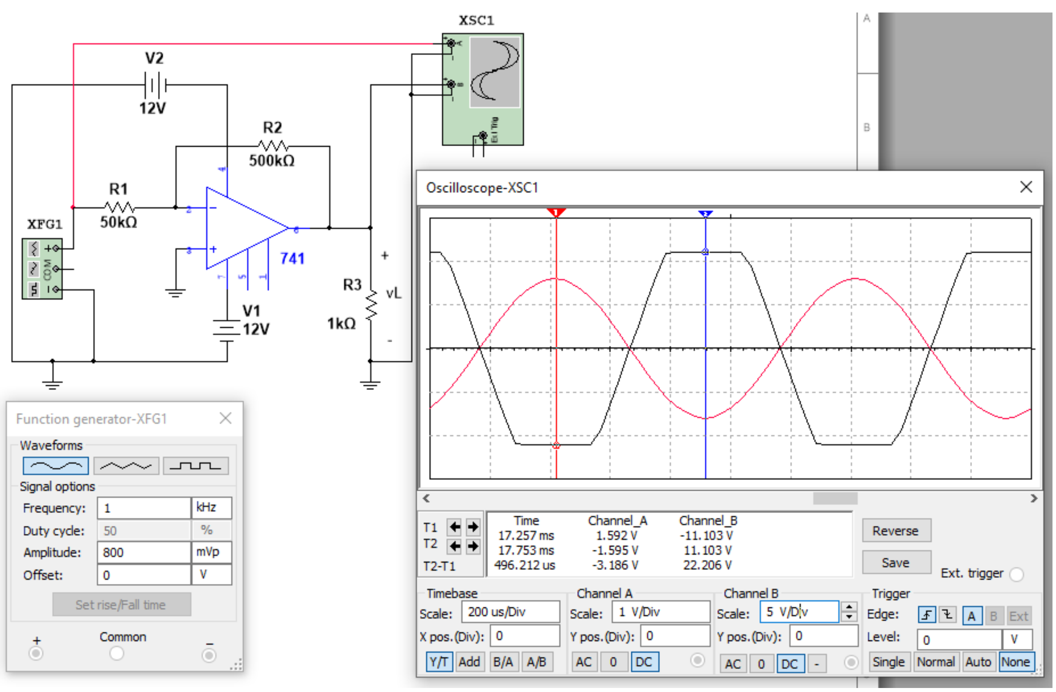


Can you explain why even though the generator says the source is a 200-mV peak sinewave, the oscilloscope reads 400mV instead?

See what happens when $V_2 = V_{EE} = 0V$

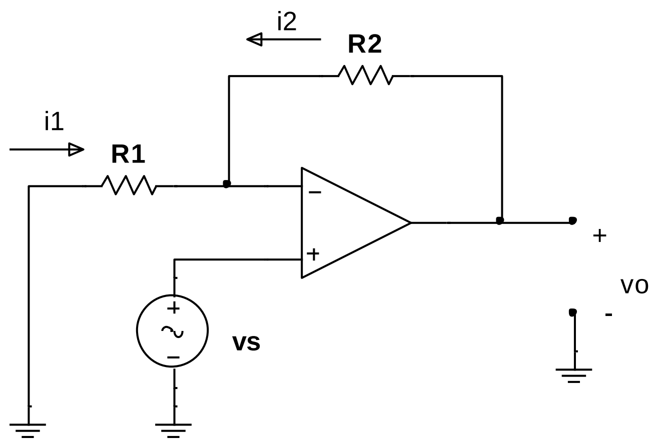


Since the lower rail is 0V, the output cannot become negative. In addition, due to internal drops related to diode junctions, the output cannot be lower than 880mV in this case.



It is obvious with the input set for 1.6Vp, the output should have had an amplitude of 16Vp but because the rails are 12 and -12V, the output is limited to 12V minus the internal drop of 880mV, which justifies the 11.1V peak reading.

5.2 Non-Inverting Amplifier



We see here that the voltage source is applied to the noninverting input while one terminal of R_1 is connected to ground. Due to negative feedback, tracking is still valid but now, $v_p = v_n = v_s$

5.2.1 Voltage Gain

Because of the zero current entering the ideal Op Amp, the two resistors form a voltage divider and v_n is the voltage across R_1.

$$v_n = \frac{R_1}{R_1 + R_2} v_o = v_p = v_s$$

$$A_v = \frac{v_0}{v_s} = \frac{R_1 + R_2}{R_1} = 1 + \frac{R_2}{R_1}$$

We can see that $A_v \geq$, and that the input signal and the output signal are in phase.

5.2.2 Input Resistance

Since the current entering the noninverting input is zero (ideal Op Amp),

$$R_{in} = \frac{v_s}{i_+} = \infty$$

5.2.3 Output Resistance

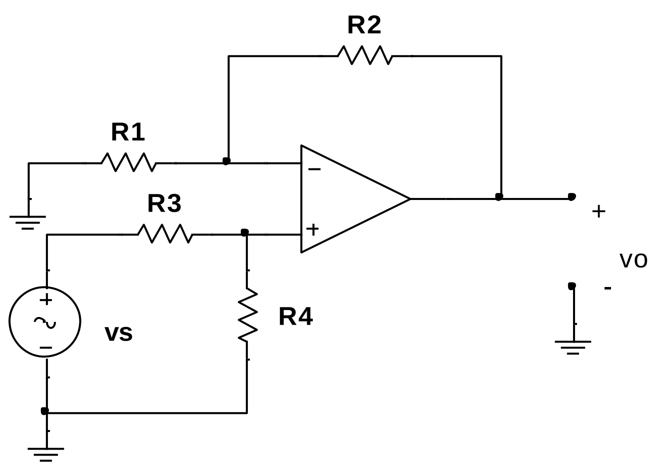
Performing similar calculations to the ones done with the inverting amplifier, we will find that

$$R_{out} = 0$$

Example

Problem:

Design a noninverting amplifier with a gain of 10, and an input resistance of 100kΩ



Solution:

Let us assume that we choose $R_3 = 10k\Omega$, and $R_4 = 90k\Omega$ to satisfy the requirements regarding the input resistance. Note that these two resistors form a voltage divider.

$$v_p = \frac{R_4}{R_3 + R_4} v_s = \frac{9v_s}{10}$$

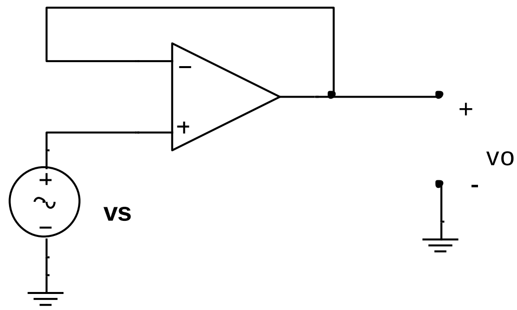
$$Choose\ R_1 = 9k\Omega, therefore\ R_2 = 91k\Omega$$

We could have chosen the last two resistors to have smaller values and still achieve the same gain. However, larger resistances will dissipate less power.

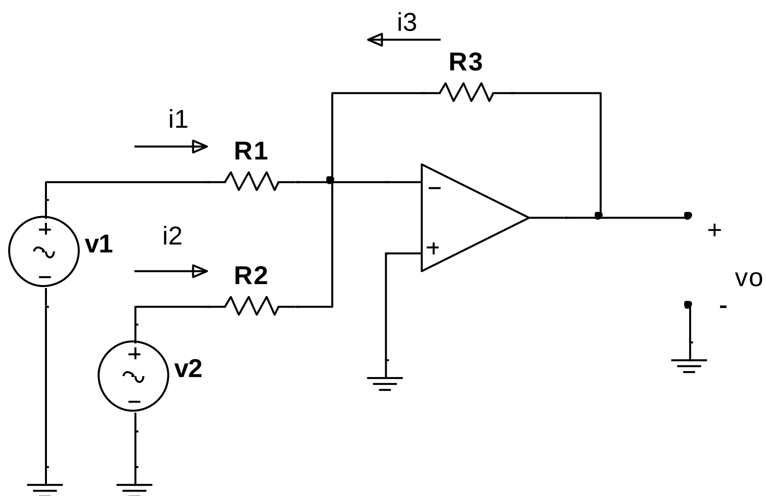
5.3 Unity Gain Buffer

If $R_2 = 0$, the gain of the noninverting amplifier becomes equal to one. Except for 0, the resistance R_1 could take any value but the most appropriate choice is infinity (absence of a resistor, less current needed, less money to spend, smaller circuit, more reliable with one component less).

This is the ideal buffer: the input resistance is infinity, the output resistance is zero, and it does not change the voltage being transferred from one stage to another.



5.4 Summing Amplifier



The inverting input is again a virtual ground due to negative feedback.

$$-v_1 + R_1 i_1 = 0$$
$$-v_2 + R_2 i_2 = 0$$
$$-v_o + R_3 i_3 = 0$$
$$i_1 = \frac{v_1}{R_1} \qquad i_2 = \frac{v_2}{R_2} \qquad i_3 = \frac{v_o}{R_3}$$

$$\text{Since} \qquad i_1 + i_2 + i_3 = 0$$

$$\frac{v_o}{R_3} = -\frac{v_1}{R_1} - \frac{v_2}{R_2}$$

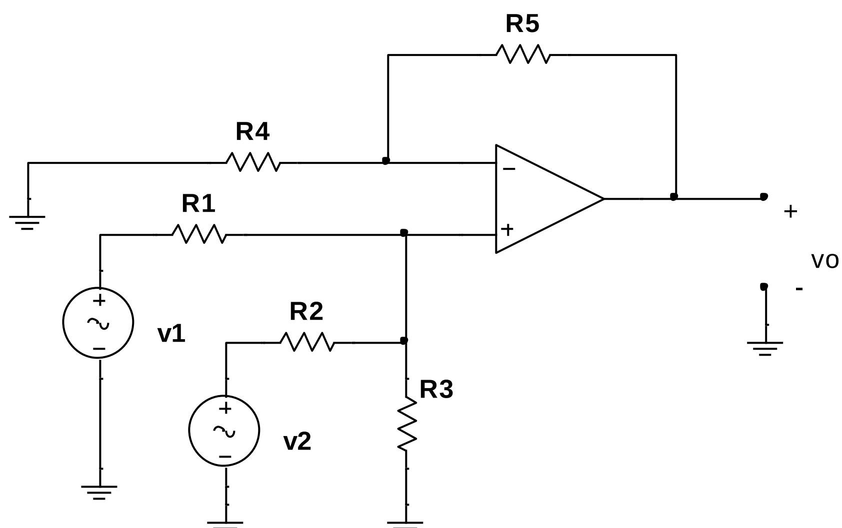
$$v_o = -\frac{R_3}{R_1} v_1 - \frac{R_3}{R_2} v_2 = a_1 v_1 + a_2 v_2$$

$$\text{where} \quad a_1 = -\frac{R_3}{R_1} \qquad a_2 = -\frac{R_3}{R_2}$$

It is clear that if $R_1 = R_2 = R_3$,

$v_o = -(v_1 + v_2)$, hence the label summing. The minus sign only affects the phase relationship, but it is still the sum of the two input signals that forms the output.

If you do not want to deal with the phase inversion, you can achieve that with a noninverting amplifier configuration. An example of such configuration is shown below.



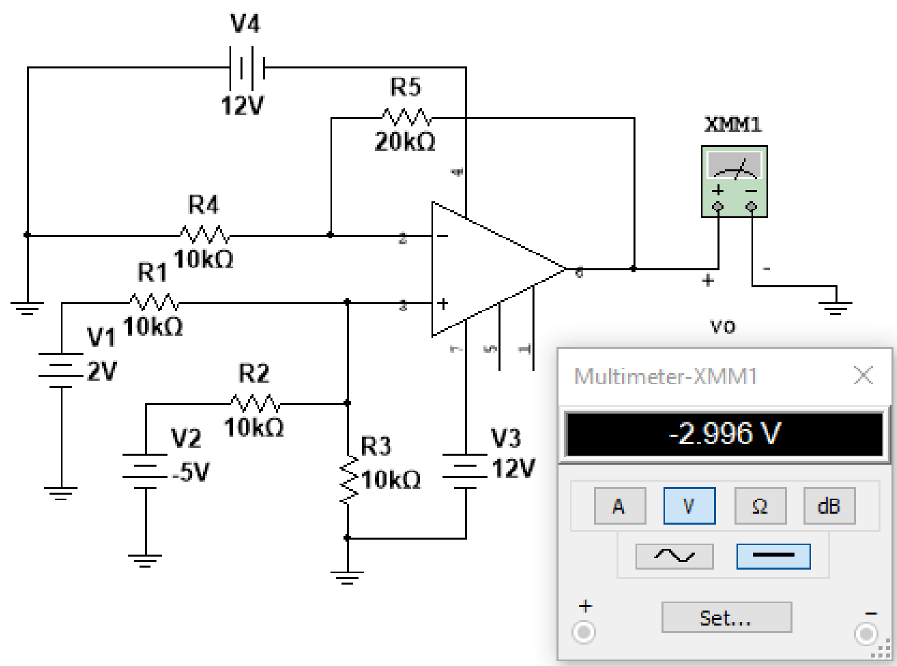
Show that the relationship between the output voltage and the two input sources is given by (hint: use superposition)

$$v_0 = \frac{R_{23}}{R_{23} + R_1}\left(1 + \frac{R_5}{R_4}\right) v_1 + \frac{R_{13}}{R_{13} + R_2}\left(1 + \frac{R_5}{R_4}\right) v_2$$

$$\text{where } R_{13} = R_1 // R_3 \quad \text{and } R_{23} = R_2 // R_3$$

In the simple case where $R_1 = R_2 = R_3$, and $R_5 = 2R_4$

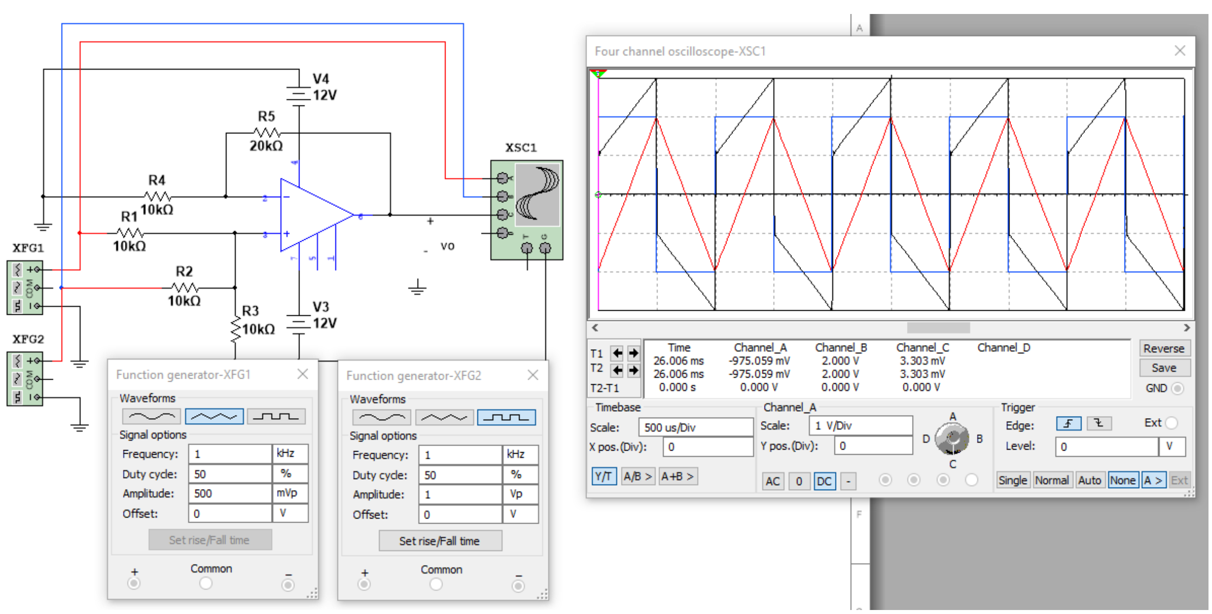
We have $v_0 = v_1 + v_2$



The measurement of the output shows that it verifies that $v_0 = v_1 + v_2$

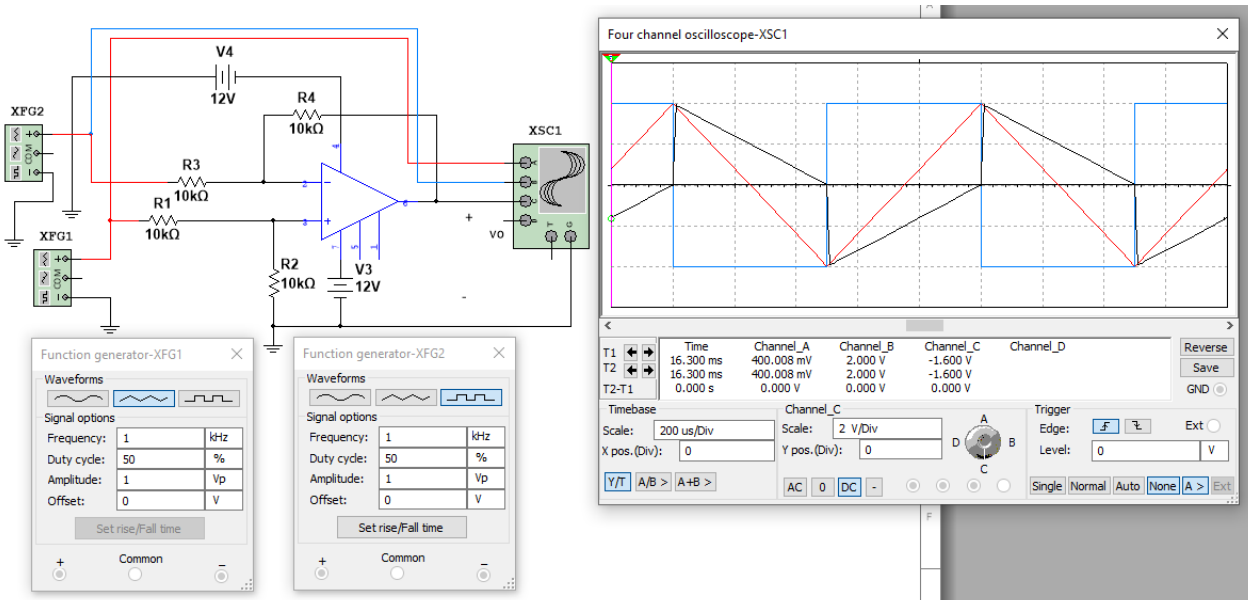
$v_0 = 2 + (-5) = -3V$ as measured by the multimeter

The same circuit is powered by a squarewave (1Vp) and a triangular wave (2Vp), And the resulting output is again the sum of the two signals. The output is displayed using a 1V/division scale. You can verify some values from the output.



5.5 Difference Amplifier

A difference amplifier or a differential subtractor amplifies the difference between two signals. An example is given below.



We will use two methods to analyze this circuit.

<u>Method1</u>:

We will find expressions for v_n and v_p and equate the two expressions because of tracking due to negative feedback

$$v_p = \frac{R_2}{R_1 + R_2} v_1$$

Using superposition

$$v_n = \frac{R_3}{R_3 + R_4} v_0 + \frac{R_4}{R_3 + R_4} v_2$$

After substitution, we have

$$v_0 = \frac{R_3 + R_4}{R_1 + R_2} \frac{R_2}{R_3} v_1 - \frac{R_4}{R_3} v_2$$

Method2:

In this case, we will use superposition. When $v_1 = 0$, the circuit is just an inverting amplifier, where $R_1//R_2$ have no effect on the value of v_0.

$$v_{01} = -\frac{R_4}{R_3} v_2$$

When $v_2 = 0$, The circuit becomes a noninverting amplifier, where v_0 goes through a voltage divider before being amplified.

$$v_{02} = \frac{R_2}{R_1 + R_2}\left(1 + \frac{R_4}{R_3}\right) v_1$$

Which leads to:

$$v_0 = \frac{R_3 + R_4}{R_1 + R_2} \frac{R_2}{R_3} v_1 - \frac{R_4}{R_3} v_2$$

For the special case where $R_1 = R_1 = R_1 = R_1 = 10k\Omega$

$v_0 = v_1 - v_2$

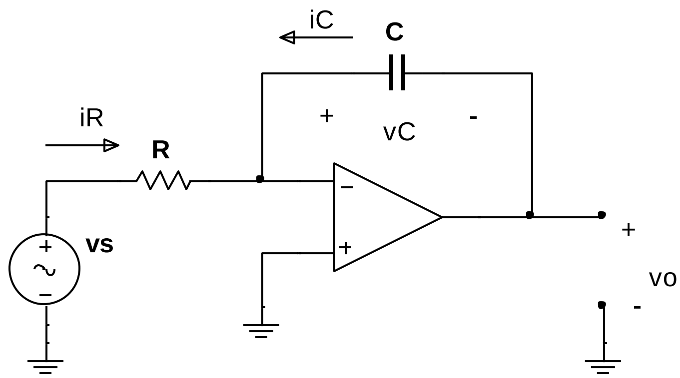
Note that v_1 is the output of generator XFG1 (2Vp triangular wave), and that v_2 is the output of generator XFG2 (2Vp square wave). One can verify that the display is a valid representation of the output voltage.

5.6 Integrator

An integrator is a circuit that will generate an output that is proportional to the integral of the input signal. Let us analyze the following circuit.

Remember that the inverting input is a virtual ground, and that no current enters the Op Amp (infinite input resistance).

In that case:



$$-v_s + Ri_R = 0$$
$$-v_0 + v_C = 0$$
$$\text{but } i_C = C\frac{dv_C}{dt} = -i_R = -\frac{v_s}{R}$$
$$\text{since } v_C = v_0, C\frac{dv_0}{dt} = -\frac{v_s}{R}$$
$$dv_0 = -\frac{1}{RC}v_s dt$$

Integrating both sides, we obtain

$$v_0(t) = -\frac{1}{RC}\int_0^t v_s(\tau)d\tau$$

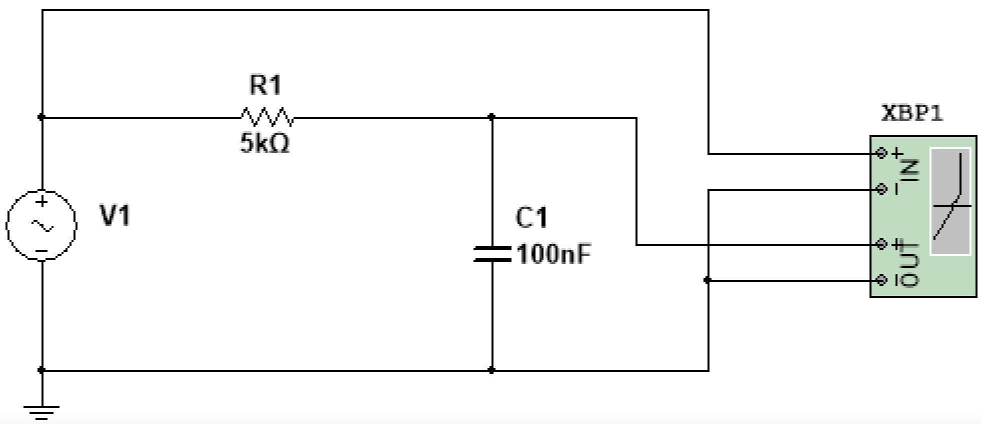
We see that the output is directly proportional to the integral of the input signal.

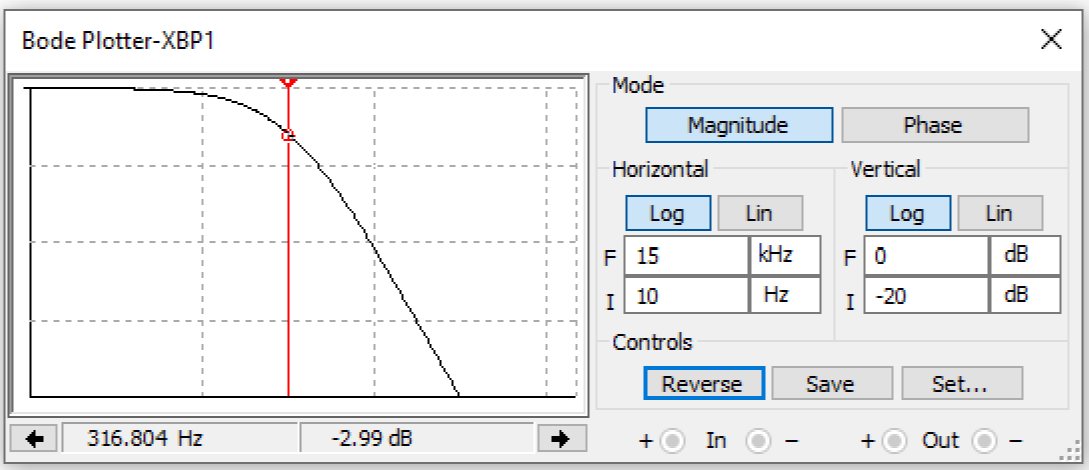
Example:

The first circuit (simple passive RC circuit, built as lowpass filter) is shown to exhibit a 3-dB cutoff frequency at

$$f_c = \frac{1}{2\pi R_1 C_1}$$

Which in this case is about 318 Hz. It is clear that around DC, the capacitor is an open circuit and the gain is 1 (0 dB), and it degrades by 20 dB/decade beyond the cutoff frequency. No gain and possible loading from any subsequent stages which will affect the frequency response makes this circuit not practical for filtering.





Example:

The next circuit is a more practical first order RC lowpass filter. It takes into account the imperfections of an Op Amp that keep it from being an ideal one (input resistance and open loop gain which are finite, an input offset voltage, and input bias currents).

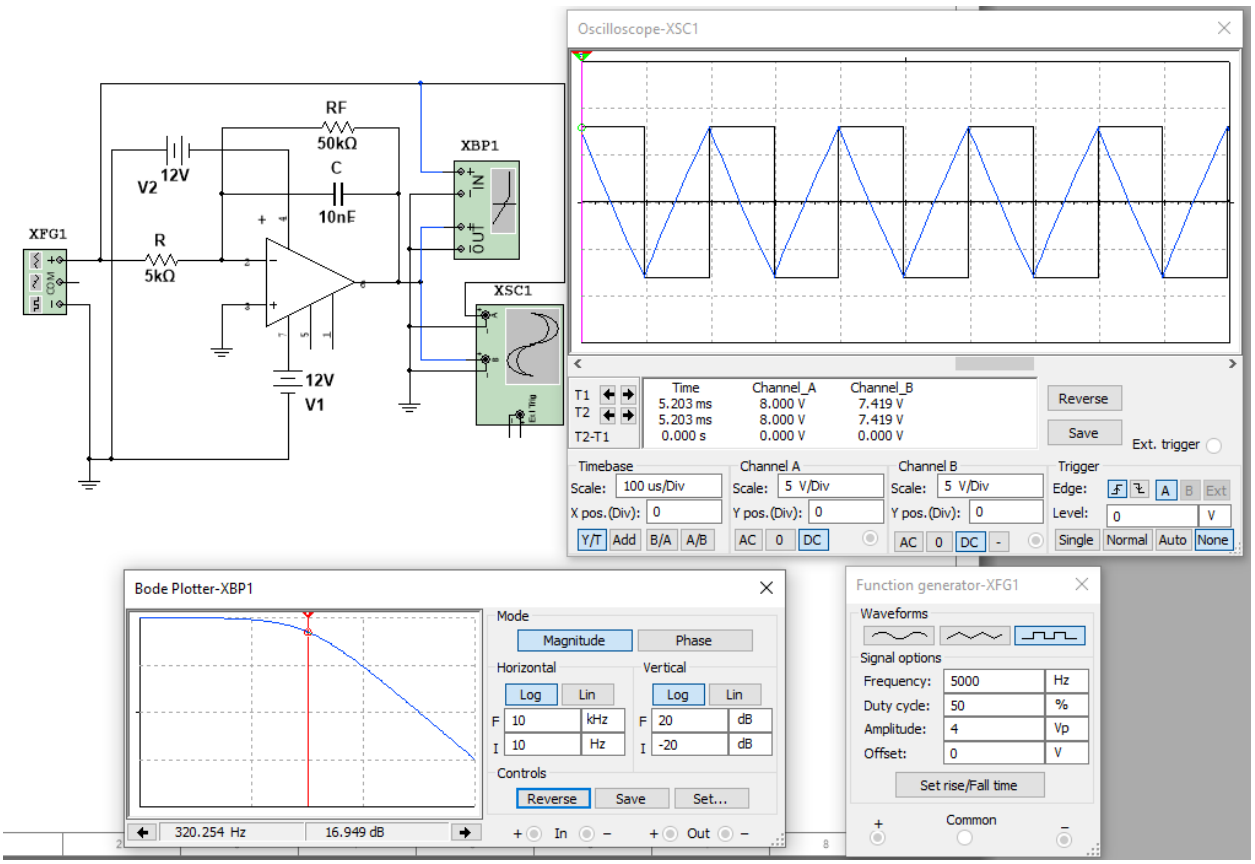
The 3-dB cutoff frequency is given by

$$f_c = \frac{1}{2\pi R_F C}$$

At DC, the simple inverting amplifier configuration has a gain of $R_F/R = 10$ (20 dB), as verified by the Bode plotter. The 3-dB cutoff frequency is verified to be near 318 Hz. In addition, we note that the frequency response has a 0-dB gain at a frequency

$$f_{0dB} \simeq \frac{1}{2\pi RC}$$

assuming that $R_F >> R$, which in this case should be and is 3180 Hz. In the range of frequencies, well beyond the 3-dB cutoff frequency (we have chosen 5000 Hz), this circuit acts like a very good integrator, with 0-DC drift.



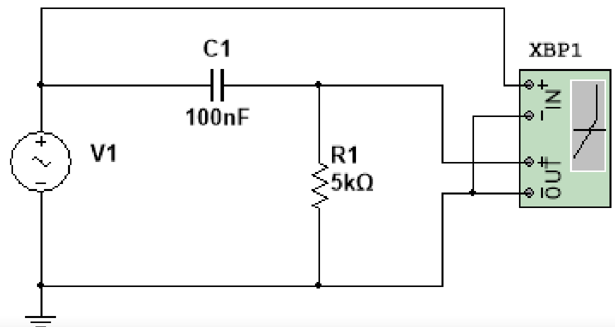
Note: for more details, see Wikipedia, Op Amp Integrator.

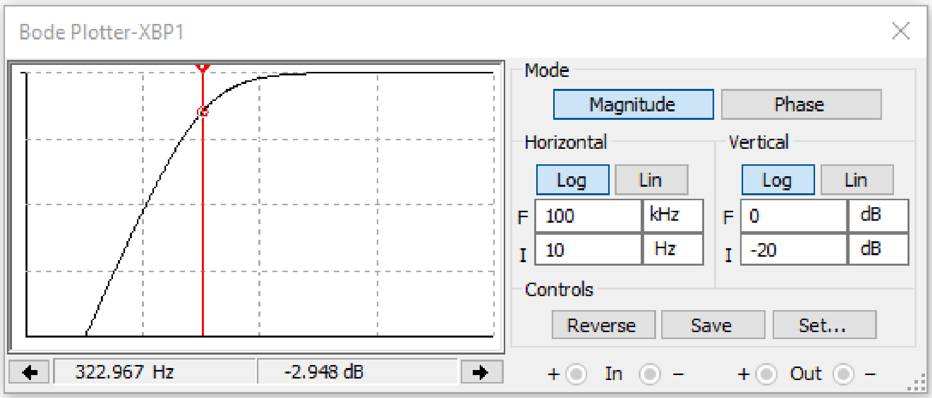
5.7 Differentiator

The differentiator is obtained from the integrator circuit by interchanging the resistor and the capacitor, R and C.

Example:

This example is about a passive RC high pass filter, whose 3-dB cutoff frequency is given by:





$$f_c = \frac{1}{2\pi R_1 C_1}$$

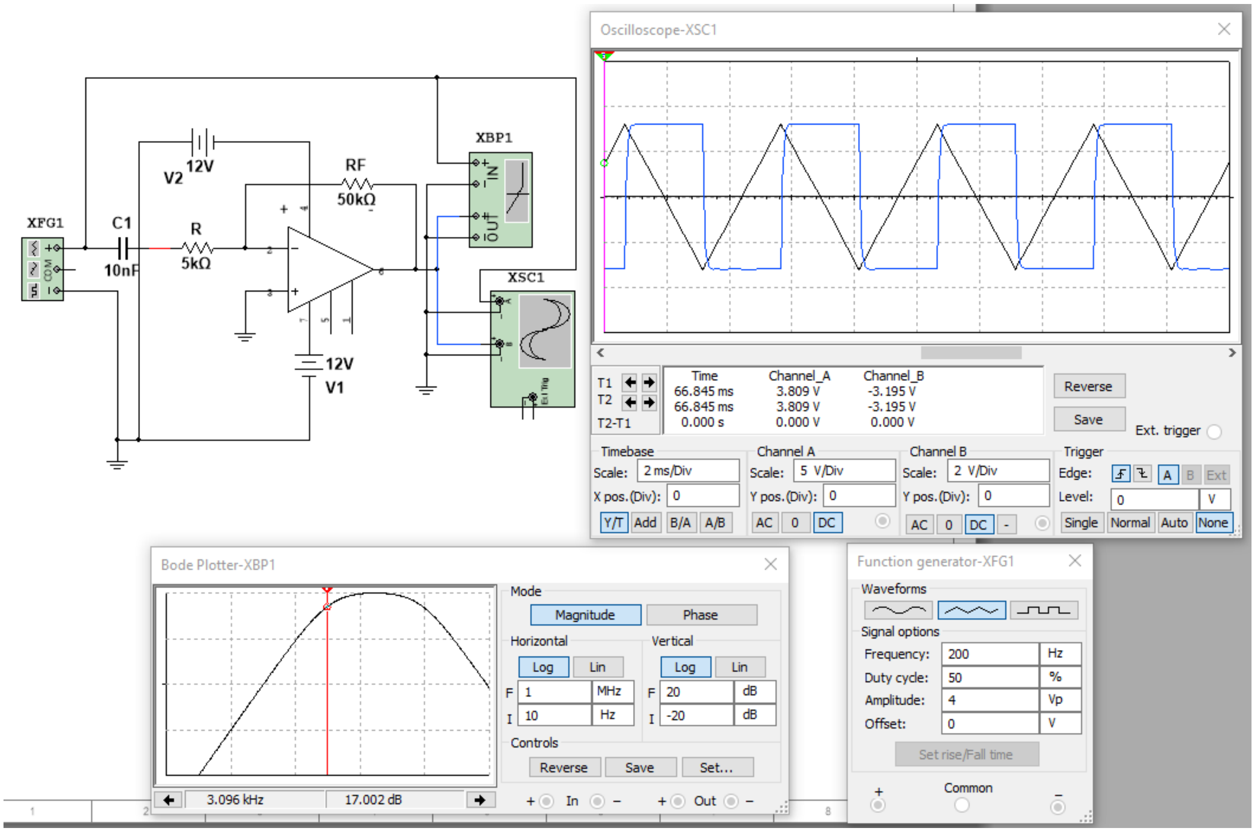
With the values used in the previous case, the cutoff frequency is the same, about 318 Hz, which can be verified through the Bode plotter.

Example:

We now demonstrate the case of a practical active high pass filter which we will utilize as a differentiator with an appropriate input signal frequency. However, the cutoff frequency was chosen to be ten times higher with a capacitor that is ten times smaller (10pF instead of 100pF). The fundamental frequency should be then 3180 Hz, and all the harmonics should all be higher. Hence if the signal is in the range of frequencies much lower than the cutoff frequency, and since the frequency response in that range is linear with a positive slope, the circuit should act like a differentiator.

This is demonstrated in the next simulation, where in addition the gain was chosen to be 10 in the range of frequencies much higher than the cutoff frequencies.

Note that we should have used a capacitor in parallel with R_F to keep the Op Amp from entering into oscillations, but we also note that the frequency response starts dipping after a passband due to the fact that the Op Amp is not ideal, acting like a bandpass filter instead. It is clear that faster Op Amps will dip at a much higher frequency.



6. Slew Rate

Slew Rate = SR = in general specified as the maximum rate of change of a voltage with respect to time, usually in V/µs. It tells how fast the variations in a signal can be and still be processed properly by the device. Otherwise, there would be delays or nonlinear effects that can contaminate the output.if the signal is a sinewave

$v(t) = A\cos(2\pi ft)$

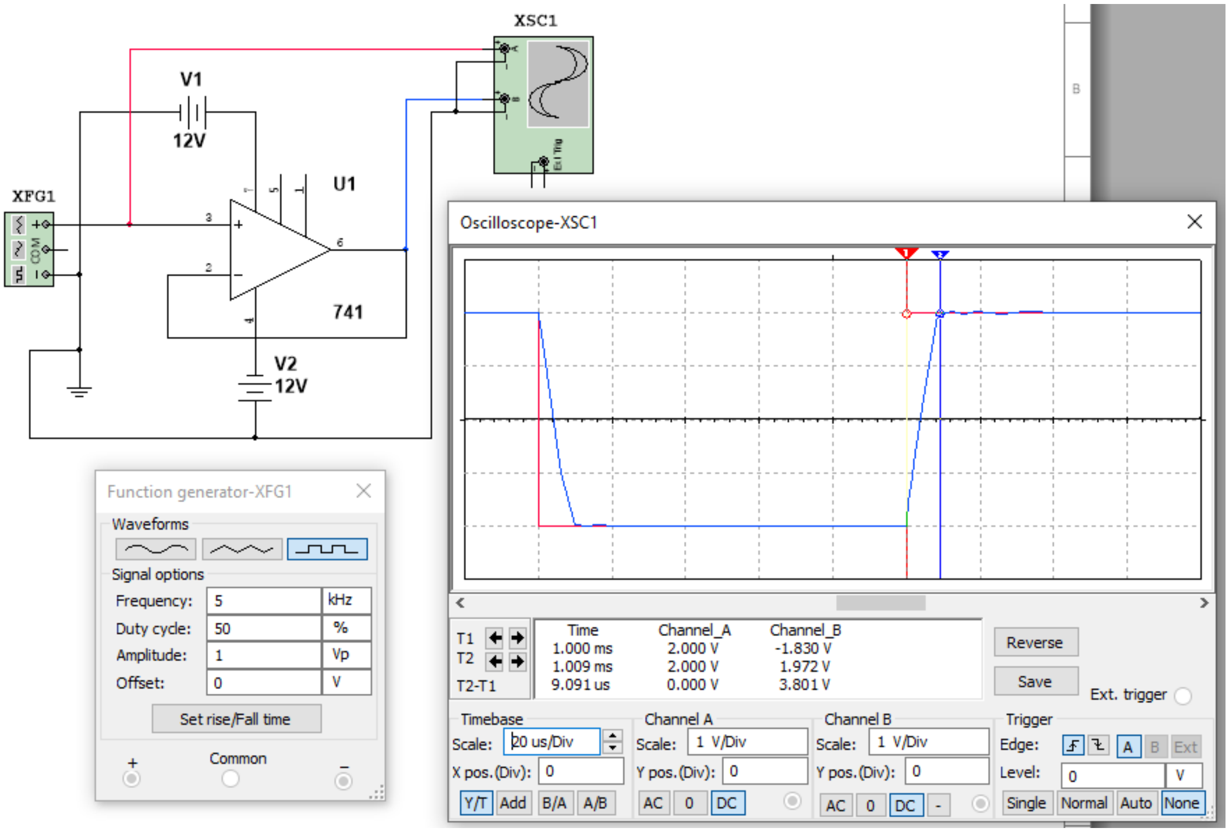
$SR = \max |dv(t)/dt|$

$= \max |2\pi f A\sin(2\pi ft)|$

$= 2\pi fA$

If the slew rate of a device exceeds $2\pi fA$, then that sinewave can be processed without any corruption.

Another way of measuring the slew rate is to inject a square wave with instantaneous transitions (or fast ones) and look at how slower the output changes from one level to the other level. The slope of the output transition is representative of the slew rate.



If we look at the simulation shown above, already at 5 kHz, the output of the 741 Op Amp takes 9µs to go from -2 to 2V, with a slew rate of 4V/9µs = 0.444V/µs.

The same value is obtained when the input frequency is increased to 50 kHz. However, the output is heavily corrupted through nonlinear effects, and the signal degrades further beyond that frequency.

Given this slew rate, and if the input were a 1Vp sinewave, then the highest frequency that can be processed appropriately is:

$2\pi f A = SR$

$f = SR/2\pi A$

$f = (0.444\ 10^{-6})/\ 2\pi$

$f = 70.7$ kHz

However, as we see in the simulation using a 50 kHz sinewave, the nonlinearities are already having an effect on the output signal. One of the reasons is the crude measurement of the slew rate, and the other is that the slew rate is representative of the maximum frequencies that can be processed without major distortion.